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LETTER TO THE EDITOR

The ($n = 0$)-component Gross–Neveu model as a description of polymers in two dimensions

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Abstract. A modified Gross–Neveu model with $n = 0$ (two fermion and two boson) components is considered in two space dimensions. This model is equivalent, in terms of perturbation theory, to the Gross–Neveu model with a $2n$-component fermion field, combined with the replica trick $n \to 0$. Using the universality hypothesis, we demonstrate that the model is related to the statistics of polymer chains. This result extends earlier investigations, obtained in a strong coupling expansion.

The ($n = 0$)-component Gross–Neveu model, defined as a lattice field theory, has been used by several authors as a description of disordered magnetic systems [1] and degenerated semiconductors [2] in two dimensions. A recent investigation of this model has pointed out that there is spontaneous symmetry breaking [3, 4] similar to that of the model with $n > 0$ [5]. Since the perturbation theory, combined with the replica trick $n \to 0$, is in disagreement with this observation [1, 3], it seems to be useful to pursue some non-perturbative investigations of this model.

A renormalisation group calculation indicates that the free-field limit of the model is unstable against a perturbation by an arbitrary small interaction term [3]. This result is not surprising, because the free-field theory obeys the symmetry mentioned above. Therefore, there is no obvious method for treating this model in terms of a perturbation theory for small interaction.

On the other hand, a strong coupling expansion is convergent, the radius of convergence is of order one and has been estimated [3]. The correlation length is always finite in the region of convergence and of the order $\log(g)$, where g is the coupling constant.

The aim of this letter is to show that the properties of the strong coupling region survive for any small positive coupling constant g, provided the universality hypothesis holds. Thus there is no 'crossover' behaviour from strong coupling to the free-field theory, but the model distinguishes only the interacting and the free-field region.

The Gross–Neveu model on the square lattice Λ is defined by the action

$$A = -\sum_{r,r' \in \Lambda} \bar{\Phi}_r \cdot \partial_{r,r'} \Phi_{r'} - g \sum_{r \in \Lambda} (\bar{\Phi}_r \cdot \Phi_r)^2 \tag{1}$$

where the propagation of the field Φ is given by

$$\partial = \sigma_1 \partial_1 + \sigma_2 \partial_2 . \tag{2}$$

σ_1, σ_2 are Pauli matrices and ∂_j is the difference operator

$$(\partial_j)_{r,r'} = \begin{cases} \pm\frac{1}{2} & \text{for } r' = r \pm e_j \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

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with the lattice unit vector e_j. According to [5], Φ is a $2n$-component fermion field. Moreover, we notice that the field theory with $2n_f = 2n$ fermion components is equivalent to a model with the same action but with $2n_f = 2(n + n_b)$ fermion and $2n_b$ boson components [6], i.e. the model depends effectively only on the difference $2(n_f - n_b)$. Therefore, it is convenient (and formally correct) to consider the ($n = 0$)-component Gross–Neveu model as a field theory with an equal number of fermion and boson components. For simplicity, we use here $n_f = n_b = 1$:

$$\Phi_r = (\Psi_{1,r}, \Psi_{2,r}, \chi_{1,r}, \chi_{2,r}) \tag{4}$$

where the fermion part Ψ is a Grassmannian and the boson part χ is a complex field.

Correlation functions or Green functions are given by expectation values

$$\langle \dots \rangle = \int \dots \exp(-A) \prod_{r \in \Lambda} \mathrm{d}\Phi_r \, \mathrm{d}\bar{\Phi}_r \tag{5}$$

with respect to the field Φ. In order to work with well defined integrations in the boson sector, we define a conjugate field in (1) as

$$\bar{\chi}_{\mu,r} = \mathrm{i} D_r (\chi_{\mu,r})^* \tag{6}$$

where D_r is a staggered field:

$$D_r = (-1)^{r_1 + r_2} \tag{7}$$

and the asterisk means complex conjugation. Such a special choice is not necessary in the fermion sector, because Ψ and $\bar{\Psi}$ are independent Grassmannians and no problem occurs for the integration. Nevertheless, it is formally convenient to have the same propagators in both sectors. Therefore, we substitute in (1)

$$\bar{\Psi}_{\mu,r} \to \mathrm{i} D_r \bar{\Psi}_{\mu,r}. \tag{8}$$

The action of (1) is then

$$A = -\mathrm{i} \sum_{r,r'} \bar{\Phi}_r \cdot (D\partial)_{r,r'} \Phi_{r'} + g \sum_r (\bar{\Phi}_r \cdot \Phi_r)^2 \tag{9}$$

with the conjugate field

$$\bar{\Phi}_r = (\bar{\Psi}_{\mu,r}, (\chi_{\mu,r})^*).$$

It should be emphasised that this representation of the ($n = 0$)-component Gross–Neveu model is identical to the replica trick in terms of a perturbation theory around $g = 0$. The factor $\mathrm{i}D_r$ can then be eliminated by a rescaling of the fields for each term of the expansion.

Due to the staggered field in the first term of (9), the inverse propagator $D\partial$ is translational invariant only on the sublattice Λ_μ for which $D_r = (-1)^\mu$. We shall show now that the ($n = 0$)-component Gross–Neveu model on Λ is related to a model of polymers on the sublattice Λ_2.

Since nearest-neighbour pairs of lattice points do not belong to the same sublattice, we may write for the non-local term of A

$$\sum_{r,r' \in \Lambda} \bar{\Phi}_r \cdot (D\partial)_{r,r'} \Phi_{r'} = \sum_{r \in \Lambda_1} [\bar{\Phi}_r \cdot (D\partial\Phi)_r + (\bar{\Phi} D\partial)_r \cdot \Phi_r]. \tag{10}$$

Furthermore, we express the interaction on Λ_1 by coupling Φ to an auxiliary Gaussian field V_r by means of the identity

$$\exp[-g(\bar{\Phi}_r\cdot\Phi_r)^2]$$

$$= (4g\pi)^{-1/2}\int \exp[\mathrm{i}(V_r+\mathrm{i}\varepsilon)\bar{\Phi}_r\cdot\Phi_r] \times\exp[-(V_r+\mathrm{i}\varepsilon)^2/4g]\,\mathrm{d}V_r \qquad \varepsilon>0 \tag{11}$$

in the expectation value (5). After interchange of the integration over Φ_r, $\bar{\Phi}_r$ and V_r, the former integration can be performed on Λ_1 now, because the action is bilinear in these fields. Although this could be done in general, we consider in the following only expectation values with respect to the sublattice Λ_2, for simplicity. Thus we obtain a model of the fields Φ on Λ_2 and V on Λ_1:

$$A_1(\Phi, V) = \sum_{r\in\Lambda_1}[\mathrm{i}(V_r+\mathrm{i}\varepsilon)^{-1}(\bar{\Phi}D\partial)_r\cdot(D\partial\Phi)_r+(V_r+\mathrm{i}\varepsilon)^2/4g]+g\sum_{r\in\Lambda_2}(\bar{\Phi}_r\cdot\Phi_r)^2 \tag{12}$$

and

$$\langle\ldots\rangle = \int\ldots\exp(-A_1)\prod_{r\in\Lambda_1}(4g\pi)^{-1/2}\,\mathrm{d}V_r\prod_{r\in\Lambda_2}\mathrm{d}\Phi_r\,\mathrm{d}\bar{\Phi}_r. \tag{13}$$

Finally, also the field V can be eliminated by integration in order to find a model which describes only a field on the sublattice Λ_2. The integral

$$\lim_{\varepsilon\downarrow 0}(4g\pi)^{-1/2}\int_{-\infty}^{\infty}\exp[-(x+\mathrm{i}\varepsilon)^2/4g+a/(x+\mathrm{i}\varepsilon)]\,\mathrm{d}x = 1+\mathrm{i}\sum_{l\geq 1}d_l a_l \tag{14}$$

with the coefficients

$$d_l = -\frac{1}{l!}\frac{2^{l-1}}{(l-1)!}(4g)^{-l/2}\begin{cases}(-1)^{(l-2)/2}\left(\dfrac{l-2}{2}\right)! & l \text{ even}\\ \mathrm{i}(-1)^{(l-1)/2}\Gamma(\frac{1}{2}l) & l \text{ odd}\end{cases} \tag{15}$$

yields a convergent series for any finite a. As a consequence, the effective action on Λ_2 is then

$$A_2(\Phi) = -\sum_{r\in\Lambda_1}\log\left(1+\mathrm{i}\sum_{l\geq 1}d_l[-\mathrm{i}(4g)^{-1/2}(\bar{\Phi}D\partial)_r\cdot(D\partial\Phi)_r]^l\right)+g\sum_{r\in\Lambda_2}(\bar{\Phi}_r\cdot\Phi_r)^2. \tag{16}$$

This rather complicated expression can be related to a simpler model by applying the universality hypothesis. The latter is based on the assumption that the qualitative properties of a model are not affected by all terms of the action in a relevant manner.

A typical parameter of a lattice model is the lattice constant. We do not expect, for instance, that the long-range behaviour (e.g. the correlation length) will change significantly under a change of this parameter. Therefore, we consider only terms of the action as relevant for the qualitative properties which do not become small with decreasing lattice constant. However, this does not mean that we take the continuum limit of the lattice model! The influence of the lattice structure on the model under consideration can be studied easily when we introduce a lattice constant $a\sqrt{2}$ $(a<1)$ instead of $\sqrt{2}$ on Λ_2. The sum $\Sigma_{r\in\Lambda_2}$ becomes on the new lattice Λ_2'

$$a^2\sum_{r\in\Lambda_2'} \tag{17}$$

while the inverse propagator rescales:

$$D\partial\to a^{-1}D\partial. \tag{18}$$

We may expand the logarithmic term of A_2 and study the effect of the shrinking lattice. The expansion term $\Sigma_{r\in\Lambda_1}[(\Phi D\partial)_r\cdot(D\partial\Phi)_r]^l$ is then related to the corresponding expression on the new lattice by

$$\sum_{r\in\Lambda_1}[(\bar{\Phi}D\partial)_r\cdot(D\partial\Phi)_r]^l=a^{\nu_l}\left(a^2\sum_{r\in\Lambda_1'}[(\bar{\Phi}'D\partial)_r\cdot(D\partial\Phi')_r]^l\right) \tag{19}$$

with the field Φ' on Λ_2' and the exponent

$$\nu_l=2(l-1). \tag{20}$$

Thus only the term with $l=1$, the leading term of the expansion, is relevant, because all other terms appear with coefficients decreasing with a decreasing lattice constant. On the other hand, the local interaction term in (16) is increasing with a shrinking lattice:

$$\sum_{r\in\Lambda_2}(\bar{\Phi}_r\cdot\Phi_r)^2=a^{-2}\left(a^2\sum_{r\in\Lambda_2'}(\bar{\Phi}_r'\cdot\Phi_r')^2\right). \tag{21}$$

After rescaling of the fields by $(\pi/4g)^{1/4}$, the action A_2 is, therefore, equivalent to

$$\bar{A}=\sum_{r\in\Lambda_1}(\bar{\Phi}D\partial)_r\cdot(D\partial\Phi)_r+\frac{4g^2}{\pi}\sum_{r\in\Lambda_2}(\bar{\Phi}_r\cdot\Phi_r)^2 \tag{22}$$

in the sense of the universality hypothesis.

The summation over Λ_1 can be performed. We find with (3) that

$$\sum_{r\in\Lambda_1}(\bar{\Phi}D\partial)_r\cdot(D\partial\Phi)_r=\sum_{r,r'\in\Lambda_2}\Delta_{r,r'}\bar{\Phi}_r\cdot\Phi_{r'} \tag{23}$$

where Δ is the lattice Laplacian

$$\Delta_{r,r'}=\begin{cases}1 & r'=r\\ -\frac{1}{4} & r'=r\pm 2e_j\\ 0 & \text{otherwise.}\end{cases} \tag{24}$$

This model describes the 'excluded volume' problem which is related to the statistics of polymer chains with a repulsive intrachain interaction [7]. Although it is common to use the replica trick in the boson sector to formulate the polymer model (i.e. there

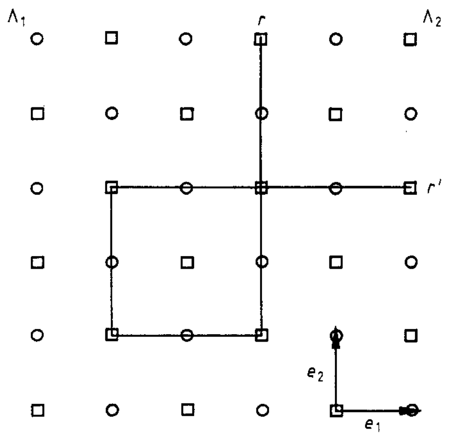

Figure 1. The square lattice Λ with the sublattices Λ_1, Λ_2, the unit vectors e_1, e_2 and a typical polymer on Λ_2.

are no fermion degrees of freedom), the fermion–boson representation of (22) is equivalent to this trick.

The effect of the repulsive self-interaction becomes apparent when we write the expectation values of the lattice field model as a statistical sum over polymer configurations. We apply for this purpose the identity (11) to the local interaction in (22). Introducing an auxiliary Gaussian field φ_r, we obtain the bilinear action

$$A_3 = \sum \Delta_{r,r'}\bar{\Phi}_r \cdot \Phi_{r'} + \mathrm{i} \sum \varphi_r \bar{\Phi}_r \cdot \Phi_r. \tag{25}$$

The Φ integration can be done explicitly and yields, for instance, for the correlation function

$$\langle \Psi_{\mu,r}\bar{\Psi}_{\mu,r'}\rangle = \langle \chi_{\mu,r}\bar{\chi}_{\mu,r'}\rangle = \overline{((\Delta+\mathrm{i}\varphi)^{-1})_{r,r'}}. \tag{26}$$

The bar denotes averaging over the Gaussian field φ. We obtain from the expansion of the inverse random matrix on the RHS, in powers of its off-diagonal elements (Id is here the unit matrix)

$$\Delta - Id \tag{27}$$

a sum over polymer chains starting at r and terminating at r':

$$\overline{((\Delta+\mathrm{i}\varphi)^{-1})_{r,r'}} = \overline{((Id+\mathrm{i}\varphi)^{-1}\textstyle\sum_{l\geqslant|r-r'|}[(Id-\Delta)(Id+\mathrm{i}\varphi)^{-1}]^l)_{r,r'}}. \tag{28}$$

The elements of the polymer chains, the monomers, have length 2 due to the lattice Laplacian (24) which connects points of distance 2. Therefore, only points r, r' with $|r_1-r_1'|$ and $|r_2-r_2'|$ zero or even can be connected by such polymers.

The RHS of (28) is a sum over statistical weights, since the matrix elements

$$Id - \Delta(=0, \tfrac{1}{4}) \tag{29}$$

are non-negative and the expectation values are positive:

$$\overline{(1+\mathrm{i}\varphi)^{-l}} = \overline{[(l-1)!]^{-1}\int_0^\infty t^{l-1}\exp[-(1+\mathrm{i}\varphi)t]\,\mathrm{d}t}$$

and after integration over φ

$$\overline{(1+\mathrm{i}\varphi)^{-1}} = [(l-1)!]^{-1}\int_0^\infty t^{l-1}\exp(-t-gt^2)\,\mathrm{d}t > 0. \tag{30}$$

The self-repulsion is given by the fact that a self-crossing chain appears with a reduced statistical weight in comparison with a non-crossing chain due to the inequality

$$\overline{(1+\mathrm{i}\varphi_r)^{-l-1}} < \overline{(1+\mathrm{i}\varphi_r)^{-1}}\;\overline{(1+\mathrm{i}\varphi_r)^{-l}}. \tag{31}$$

Furthermore, the boundedness of

$$\tau = \overline{(1+\mathrm{i}\varphi_r)^{-1}} < 1 \tag{32}$$

and its asymptotic behaviour for $g \sim 0$

$$\tau \sim 1 - 2g^2/\pi$$

ensure the existence of the series in (28). There are at most 4^R different polymers consisting of R monomers, since each monomer can be arranged in four different directions on the square lattice. Each contributes a factor $\frac{1}{4}$ according to (29). Hence the RHS of (28) is bounded from above by

$$\sum_{l\geqslant|r-r'|} \tau^l = \frac{\tau^{|r-r'|}}{(1-\tau)} \tag{33}$$

i.e. the correlation function decays exponentially for any $g > 0$. A non-exponential decay can occur if we add a negative mass term

$$m \sum_{r \in \Lambda_2} \bar{\Phi}_r \cdot \Phi_r \qquad 0 > m = \mathrm{o}(g) \tag{34}$$

to the action $\bar{A}$ of (22). However, such a mass term would correspond to an imaginary mass of the Gross–Neveu model (1), a case which has not been considered on a physical basis, so far.

The interpretation of the $(n = 0)$-component Gross–Neveu model in terms of classical statistics supports earlier speculations, based on the renormalisation group theory and strong coupling expansion [3], that this model cannot describe critical behaviour of disordered spin systems in two dimensions.

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